Vacuum Einstein metrics with bidimensional Killing leaves.*
II-Global aspects.

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Abstract

A formalism (ζ-*complex analysis*), allowing one to construct global Einstein metrics by matching together local ones described in the papers [Phys. Lett. **B** 513(2001)142-146; Diff. Geom. Appl. **16**(2002)95-120], is developed. With this formalism the singularities of the obtained metrics are described naturally as well. *Subj. Class.:* Differential Geometry, General Relativity. *Keywords:* Einstein metrics, Killing vectors. *MS classification:* 53C25

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1 Introduction

In the previous paper [4] we described local form of vacuum Einstein metrics that admit a Killing algebra $\mathcal{G}$, such that

I the distribution $\mathcal{D}$, generated by the vector fields belonging to $\mathcal{G}$, is bidimensional

II the distribution $\mathcal{D}^{\perp}$, orthogonal to $\mathcal{D}$, is completely integrable and transversal to $\mathcal{D}$.

In this paper we answer the natural question: how to match together these local metrics, in order to get *global nonextendable* ones. An important common peculiarity of the considered local metrics is that they all are fibered over ζ*-complex curves* (see next section). This fact allows one to reduce the problem to a much simple one in zeta-complex analysis. Moreover, singularities of Einstein metrics, which are inevitable according to Hawking's theorem, may be described with this technique in a simple transparent manner. The same technique offers also various possibilities in manipulating with already known Einstein metrics to get new ones. For instance, just by "extracting the square roots" from the Schwarzschild metric one discovers two "parallel universes" generated by two "parallel stars". Geometrical properties of solutions described in the paper will be discussed with more details separately. Here, we will limit ourselves to a few examples (section 4) illustrating some aspects of our approach which generalizes naturally to several situations as, for instance, *cosmological Einstein metrics* satisfying assumptions I and II (work in progress). In this paper we continue to use terminological and notational conventions, adopted in [4], without a special mentioning.

2 Global solutions

From the local analytic description of Ricci-flat metrics, given in [4] (section *5* and *6)*, it is not immediately evident whether they are pair-wise different or not. Here we will give them a coordinate-free description, so that it becomes clear what variety of different geometries, in fact, is obtained. We will see that with any of the found solutions a pair, consisting of a ζ*-complex curve* $\mathcal{W}$ and a ζ*-harmonic* function u on it, is associated. If two solutions are equivalent, then the corresponding pairs, say $(\mathcal{W}, u)$ and $(\mathcal{W}', u')$, are related by an invertible ζ-holomorphic map $\Phi : (\mathcal{W}, u) \longrightarrow (\mathcal{W}', u')$ such that $\Phi^{*}(u') = u$. Roughly speaking, the *moduli space* of the obtained geometries is surjectively mapped on the *moduli space* of the pairs $(\mathcal{W}, u)$. Further parameters,

distinguishing the metrics we are analyzing, are given below. Before that, however, it is worth to underline the following common peculiarities of these metrics.

Proposition 1. *Orthogonal leaves are totally geodesic and possess a non trivial Killing field. Geodesic flows, corresponding to metrics, admitting 3-dimensional Killing algebras, are non-commutatively integrable.*

Proof. The first assertion follows from the fact that the metrics have, in the adapted coordinates, a block diagonal form whose upper block does not depend on the last two coordinates. The existence of a non trivial Killing field is obvious from the description of model solution given in next section. For what concerns geodesic flows, they are integrated explicitly for model solution in next section, and the general result follows from the fact that any solution is a pullback of a model one. □

Solutions of the Einstein equations found in [4] (section *5*) manifest an interesting common feature. Namely, each of them is determined completely by a choice of

1) a solution of the wave, or the Laplace equation, depending on the sign of $\det \mathbf{F}$;

and either by

2′) a choice of the constant A and one of the branches (see fig.1 in section *2.2*) of the tortoise equation

$$\beta + A \ln |\beta - A| = u, \tag{1}$$

if $h_{22} \neq 0$, or by

2″) a choice of a solution of one of the two equations

$$\left[\mu\left(\partial_y^2 - \partial_x^2\right) + \mu_y \partial_y - \mu_x \partial_x\right] w = 0 \qquad \Box\mu = 0, \tag{2}$$

$$\left[\mu\left(\partial_y^2 + \partial_x^2\right) + \mu_y \partial_y + \mu_x \partial_x\right] w = 0 \qquad \triangle\mu = 0, \tag{3}$$

according to the sign of $\det \mathbf{F}$, in the case $h_{22} = 0$.

Solutions we are analyzing in the paper have a natural *fibered structure* with the Killing leaves as fibers. The wave and Laplace equations, mentioned above in 1), are in fact defined on the bidimensional manifold $\mathcal{W}$ which parameterizes the Killing leaves. These leaves themselves are

bidimensional Riemannian manifolds and, as such, are geodesically complete. For this reason the problem of the extension of local solutions, found in [4] (section *5*)*,* is reduced to that of the extension of the base manifold $\mathcal{W}$. Such an extension should carry a geometrical structure that gives an intrinsic sense to the notion of the wave or the Laplace equation and to equations (2) and (3) on it. A brief description of how this can be done is the following.

2.1 $\zeta-$complex structures

Recall that there exist three different isomorphism classes of bidimensional commutative unitary algebras. They are

$$\mathbb{C} = \mathbb{R}[x] / \left(x^2+1\right), \qquad \mathbb{R}_{(2)} = \mathbb{R}[x] / \left(x^2\right), \qquad \mathbb{R}\oplus\mathbb{R} = \mathbb{R}[x] / \left(x^2-1\right).$$

Elements of this algebra can be represented in the form $a+\zeta b$, $a, b \in \mathbb{R}$, with $\zeta^2 = -1, 0$, or 1, respectively. For a terminological convenience we will call them ζ *-complex numbers*. Of course, ζ-complex numbers for $\zeta^2 = -1$ are just ordinary complex numbers. Furthermore, we will use the unifying notation $\mathbb{R}^2_\zeta$ for the algebra of ζ-complex numbers. For instance $\mathbb{C} = \mathbb{R}^2_\zeta$ for $\zeta^2 = -1$.

Remark 1. In the literature one can find various alternative names for elements of these algebras. For instance, *dual* numbers for $\mathbb{R}_{(2)}$, *double* or *Zarissky* numbers for $\mathbb{R}\oplus\mathbb{R}$, etc.

In full parallel with ordinary complex numbers, it is possible to develop a ζ*-complex analysis* by defining ζ*-holomorphic* functions as $\mathbb{R}^2_\zeta$-valued differentiable functions of the variable $z = x+\zeta y$. Just as in the case of ordinary complex numbers, the function $f(z) = u(x,y)+\zeta v(x,y)$ is ζ-holomorphic *iff* the ζ-Cauchy-Riemann conditions hold:

$$u_x = v_y, \qquad u_y = \zeta^2 v_x \tag{4}$$

The compatibility conditions of the above system requires that both u and v satisfy the ζ*-Laplace equation*, that is

$$-\zeta^2 u_{xx} + u_{yy} = 0, \qquad -\zeta^2 v_{xx} + v_{yy} = 0.$$

Of course, the ζ-Laplace equation reduces for $\zeta^2 = -1$ to the ordinary Laplace equation, while for $\zeta^2 = 1$ to the wave equation. The operator $-\zeta^2\partial_x^2 + \partial_y^2$ will be called the ζ*-Laplace operator.*

Definition 1. (i) A ζ-complex structure on $\mathcal{W}$ is an endomorphism $J : \mathcal{D}(\mathcal{W}) \to \mathcal{D}(\mathcal{W})$ of the $C^\infty(\mathcal{W})$ module $\mathcal{D}(\mathcal{W})$ of all vector fields on $\mathcal{W}$, with $J^2 = \zeta^2\mathbf{I}$, $J \neq 0, \mathbf{I}$, and vanishing

Nijenhuis torsion, i.e., $[J,J]^{FN}=0$, where $[,]^{FN}$ stands for the Frölicher-Nijenhuis bracket. (ii) A bidimensional manifold $\mathcal{W}$ supplied with a *ζ-complex structure* is called a *ζ-complex curve.*

Obviously, for $\zeta^2=-1$ a ζ-complex curve is just an ordinary *1*-dimensional complex manifold (curve). By using the endomorphism J the ζ-Laplace equation can be written intrinsically as

$$d\left(J^{*}du\right)=0,$$

where $J^{*}:\Lambda^{1}\left(\mathcal{W}\right)\longrightarrow\Lambda^{1}\left(\mathcal{W}\right)$ is the *adjoint to J endomorphism* of the $C^{\infty}\left(\mathcal{W}\right)$ module of 1-forms on $\mathcal{W}$. Given a bidimensional smooth manifold $\mathcal{W}$, an atlas $\{(U_i,\Phi_i)\}$ on $\mathcal{W}$ is called *ζ-complex iff*

i) $\Phi_i:U_i\longrightarrow\mathcal{W},\quad U_i$ is open in $\mathbb{R}_{\zeta}^{2}$,

ii) the transition functions $\Phi_j^{-1}\circ\Phi_i$ are ζ-holomorphic.

Two ζ-complex atlases on $\mathcal{W}$ are said to be *equivalent* if their union is again a ζ-complex atlas.

A class of ζ-complex atlases on $\mathcal{W}$ supplies, obviously, $\mathcal{W}$ with a ζ-complex structure. Conversely, given a ζ-complex structure on $\mathcal{W}$ there exists a ζ-complex atlas on $\mathcal{W}$ inducing this structure. Charts of such an atlas will be called *ζ-complex coordinates* on the corresponding ζ-complex curve. In ζ-complex coordinates the endomorphism J and its adjoint J^{*} are described by the relations

$$J\left(\partial_x\right)=\partial_y,\quad J\left(\partial_y\right)=\zeta^2\partial_x$$
$$J^{*}\left(dx\right)=\zeta^2dy,\quad J^{*}\left(dy\right)=dx.$$

If $\zeta^2\neq 0$, the functions u and v in the Eq. (4) are said to be *conjugate*. Alternatively, a ζ-complex curve can be regarded as a bidimensional smooth manifold supplied with a specific atlas whose transition functions

$$(x,y)\longmapsto(\xi(x,y),\eta(x,y))$$

are subjected to ζ-Cauchy-Riemann relations (4).

Remark 2. It is not difficult to see that for $\zeta^2=1$ a ζ -complex structure on a bidimensional manifold is completely determined by its characteristic distribution, *i.e.*, by two *1*-dimensional distributions composed of characteristic vectors of the corresponding ζ-Laplace equation, and conversely.

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As it is easy to see, the ζ-Cauchy-Riemann relations (4) imply that

$$\partial_\eta^2 - \zeta^2 \partial_\xi^2 = \frac{1}{\xi_x^2 - \zeta^2 \xi_y^2} \left(\partial_y^2 - \ \zeta^2 \partial_x^2 \right),$$

and also

$$\mu \left(\partial_\eta^2 - \zeta^2 \partial_\xi^2 \right) + \mu_\eta \partial_\eta - \zeta^2 \mu_\xi \partial_\xi = \frac{1}{\xi_x^2 - \zeta^2 \xi_y^2} \left[\mu \left(\partial_y^2 - \ \zeta^2 \partial_x^2 \right) + \mu_y \partial_y - \zeta^2 \mu_x \partial_x \right].$$

This shows that equation (2) (respectively, (3)) is well-defined on a ζ-complex curve with $\zeta^2 = 1$ (respectively, $\zeta^2 = -1$). The manifestly intrinsic expression for these equations is

$$d \left(\mu J^* dw \right) = 0.$$

We will refer to it as the *μ-deformed ζ-Laplace equation.* A solution of the ζ-Laplace equation on $\mathcal{W}$ will be called *ζ-harmonic*. We can see that in the case $\zeta^2 \neq 0$ the notion of *conjugate ζ-harmonic function* is well defined on a ζ-complex curve. In addition, notice that the metric field $d\xi^2 - \zeta^2 d\eta^2$, η being ζ-conjugate with ξ, is canonically associated with a ζ-harmonic function ξ on $\mathcal{W}$. A map $\Phi : \mathcal{W}_1 \longrightarrow \mathcal{W}_2$ connecting two ζ-complex curves will be called *ζ -holomorphic* if $\varphi \circ \Phi$ is locally ζ-holomorphic for any local ζ-holomorphic function φ on $\mathcal{W}_2$. Obviously, if Φ is ζ-holomorphic and u is a ζ -harmonic function on $\mathcal{W}_2$, then $\Phi^* (u)$ is ζ -harmonic on $\mathcal{W}_1$.

- The *standard ζ-complex curve* is $\mathbb{R}_\zeta^2 = \{(x + \zeta y)\}$, and the *standard ζ-harmonic function* on it is given by x, whose conjugated is y.

The pair $\left(\mathbb{R}_\zeta^2, x \right)$ is *universal* in the sense that for a given ζ-harmonic function u on a ζ-complex curve $\mathcal{W}$ there exists a ζ-holomorphic map $\Phi : \mathcal{W} \longrightarrow \mathbb{R}_\zeta^2$ defined uniquely by the relations $\Phi^* (x) = u$ and $\Phi^* (y) = v$, v being conjugated with u.

2.2 Global properties of solutions

The above discussion shows that any global solution, that can be obtained by matching together local solutions found in [4] (section *5*), is a solution whose base manifold is a ζ-complex curve $\mathcal{W}$ and which corresponds to a ζ-harmonic function u on $\mathcal{W}$. A solution of Einstein equations corresponding to $\mathcal{W} \subseteq \mathbb{R}_\zeta^2$, $u \equiv x$ will be called a *model*. Notice that there exist various model solutions due to various options in the choice of parameters appearing in $2'$) and $2''$) at the beginning of this section. An important role played by the model solutions is revealed by the following assertion:

Proposition 2. *Any solution of the Einstein equation which can be constructed by matching together local solutions of [4] (section 5) is the pullback of a model solution via a ζ-holomorphic map from a ζ-complex curve to $\mathbb{R}^2_\zeta$.*

Proof. It follows directly from the fact that any ζ-harmonic function on a ζ-complex curve $\mathcal{W}$ is the pullback of x by a suitable ζ-holomorphic map $\mathcal{W} \longrightarrow R^2_\zeta$. □

Now we can resume in a systematic way the results obtained in the first part [4]. We distinguish between the two following qualitatively different cases:

I metrics admitting a normal *3*-dimensional Killing algebra with bidimensional leaves;

II metrics admitting a normal bidimensional Killing algebra that does not *extend* to a larger algebra having the same leaves and whose distribution orthogonal to the leaves is integrable.

It is worth mentioning that the distribution orthogonal to the Killing leaves is automatically integrable in *Case I* (proposition *16* of [4], section *7*). Also in *Case II* the bidimensionality of the Killing leaves is guaranteed by proposition *2* of [4]. Any Ricci-flat manifold (M, g), we are analyzing, is fibered over a ζ-complex curve $\mathcal{W}$

$$\pi : M \longrightarrow \mathcal{W}$$

whose fibers are the Killing leaves and as such are bidimensional Riemann manifolds of constant Gauss curvature. Below, we shall call π the *Killing fibering* and assume that its fibers are *connected and geodesically complete*. Therefore, maximal (*i.e., non-extendible*) Ricci-flat manifolds, of the class we are analyzing in the paper, are those corresponding to maximal (*i.e., non-extendible*) pairs $(\mathcal{W}, u)$, where $\mathcal{W}$ is a ζ-complex curve and u is ζ-harmonic function on $\mathcal{W}$.

2.2.1 *Case I*

Here the Killing algebra $\mathcal{G}$ is isomorphic to one of the following: $so\,(3)$, $so\,(2,1)$, $\mathcal{K}il\left(dx^2 \pm dy^2\right)$, $\mathcal{A}_3$ (see section *7* in [4]), and the Killing fibering splits in a canonical way into the Cartesian product. This product structure can be interpreted as a flat connection in π, determined uniquely by the requirements that its parallel sections are orthogonal to the Killing leaves and the parallel transports of fibers are their conform equivalences (with respect to the induced metrics). In that sense one can say that the Killing fibering is supplied canonically with a conformally flat connection.

This is a geometrically intrinsic way to describe the Killing fibering. A discrete isometry group acting freely on a fiber of the Killing fibering can be extended fiber-wise to the whole of M due to the canonical product structure mentioned above. Conversely, a locally isometric covering $\psi : \widetilde{S} \longrightarrow S$ of a fiber S allows to construct a locally isometric covering $\widetilde{M} \longrightarrow M$ which copies ψ fiber by fiber. So any homogeneous bidimensional Riemannian manifold can be realized as typical fiber of a Killing fibering. Denote by (Σ, g_Σ) a homogeneous bidimensional Riemannian manifold, whose Gauss curvature $K(g_\Sigma)$, if different from zero, is normalized to ± 1. Denote by $(\mathcal{W}, u)$ the pair constituted by a ζ-complex curve $\mathcal{W}$ and a ζ-harmonic function u on $\mathcal{W}$. Denote also by π_1 (respectively, π_2) the natural projection of $M = \mathcal{W}\Sigma$ on $\mathcal{W}$ (respectively, on Σ). Then, the above data determine the following Ricci- flat manifold (M, g) with

$$M = \mathcal{W}\Sigma, \qquad g = \pi_1^*\left(g_{\{u\}}\right) + \pi_1^*\left(\beta^2\right)\pi_2^*\left(g_\Sigma\right) \tag{5}$$

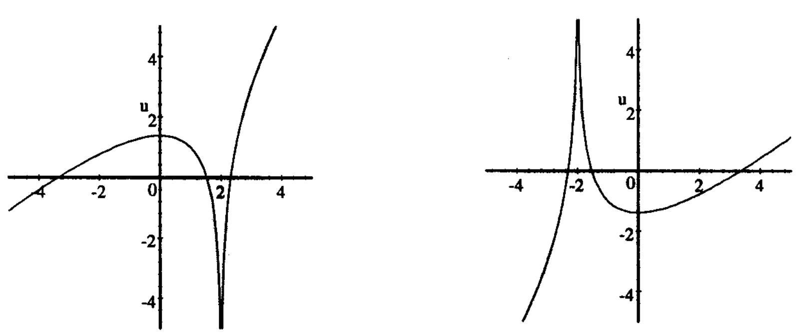
where $\beta = \beta(u)$ is implicitly determined by u via the equation

$$\beta + A \ln|\beta - A| = u \tag{6}$$

and

$$g_{\{u\}} = \epsilon \frac{\beta - A}{\beta}\left(du^2 - \ \zeta^2 dv^2\right) \tag{7}$$

A being an arbitrary constant and $\epsilon = \pm 1$. Only in the case $A = 0$ the equation (6) determines the function $\beta(u)$ uniquely: $\beta \equiv u$ and g is flat. If $A \neq 0$ the graph of the left hand side of Eq.(6) is as follows



$u = \beta + 2\ln|\beta - 2|$ $\qquad$ $u = \beta - 2\ln|\beta + 2|$

*Fig.*1

Thus, one can see that for $A \neq 0$ there are up to three possibilities for $\beta = \beta(u)$ that correspond to the intervals of monotonicity of $u(\beta)$. For instance, for $A > 0$ these are $]-\infty, 0[$, $]0, A[$, and

$]A,\infty[$. In these regions the metric (7) is regular and has some singularities along the curves $\beta = 0$ and $\beta - A = 0$.

Remark 3. Solutions characterized by A_0, $u_{0,}\, \beta_0\,(u_0)$ are globally diffeomorphic, *via* $x \leftrightarrow -x$, to solutions characterized by $A_1 = -A_0$, $u_1 = -u_{0,}\, \beta_1\,(u_1) = -\beta_0\,(-u_1)$.

Theorem 3. *Any Ricci-flat* 4*-metric admitting a normal Killing algebra isomorphic to* $so\,(3)$ *or* $so\,(2,1)$ *with bidimensional leaves, is of the form (5).*

Proof. For what concerns the algebras isomorphic to $so\,(2,1)$ it is sufficient to observe that solutions obtained in [4] (section *5*) (the case $h_{22} \neq 0$) are locally of that form. As for the Killing algebras isomorphic to $so\,(3)$, it is a direct consequence of the results of ([4] section 7). □

In the case of normal Killing algebras isomorphic to $\mathcal{K}il\left(dx^2 \pm dy^2\right)$ consider Ricci-flat manifolds M of the form

$$M = \mathcal{W}\pm, \qquad g = \pi_1^*\left(g_{[u]}\right) + \pi_1^*\,(u)\,\pi_2^*\,(g_\Sigma) \tag{8}$$

where (Σ, g_Σ) is a flat bidimensional manifold and

$$g_{[u]} = \epsilon \frac{1}{\sqrt{u}}\left(du^2 - \zeta^2 dv^2\right)$$

with $\epsilon = \pm 1$.

Theorem 4. *Any Ricci-flat* 4*-metric, admitting a normal Killing algebra isomorphic to* $Kil\left(dx^2 \pm dy^2\right)$*, and with bidimensional Killing leaves, is either of the form (8) or flat.*

Proof. In the case of a Killing algebra isomorphic to $\mathcal{K}il\left(dx^2 - dy^2\right)$ it is just an interpretation of propositions 10 and 13 of [4]. For instance, proposition 13 deals with the case $\zeta^2 = -1$. Since in our case w is constant, the lower block of $\mathbf{M}_C\,(g)$ has obviously the form $\pi_1^*\,(u)\,\pi_2^*\,(g_\Sigma)$ with $u = D\varphi + B$. Even more, if $D \neq 0$ the metric given by $\mathbf{M}_C\,(g)$ is manifestly of the form (8). If $D = 0$, then the upper block of $\mathbf{M}_C\,(g)$ can be brought to the form $\gamma\left(d\varphi^2 + d\psi^2\right)$, where φ and ψ are conjugated harmonic functions and $\gamma = \varepsilon\sqrt{|B|}$. This shows that the whole metric is flat. A similar argument can be applied to the case $h_{22} = 0$ of section *6* of [4] to conclude the proof for the case $\mathcal{K}il\left(dx^2 + dy^2\right)$. □

2.2.2 *Case II*

In this case, a coordinate-free description of global solutions, obtained in a local form in section *5* of [4], is as follows. Let $(\mathcal{W}, u)$ be as before, and w be a solution of the equation $d\left(u\, J^*\left(dw\right)\right) = 0$. Consider the flat indefinite Euclidean plane $\left(\mathbb{R}^2,\ d\xi^2 - d\eta^2\right)$ introduced at the end of section *3* of [4]. Then the direct product $M = \mathcal{W}\mathbb{R}^2$ can be supplied with the following Ricci-flat metric

$$g = \pi_1^*\left(g_{[u]}\right) + \pi_1^*\left(u\right)\pi_2^*\left(d\xi^2 - d\eta^2\right) + \pi_1^*\left(uw\right)\pi_2^*\left(\left(\frac{d\xi - d\eta}{\xi - \eta}\right)^2\right), \tag{9}$$

where $\pi_1 : M = \mathcal{W}\mathbb{R}^2 \to \mathcal{W}$ and $\pi_2 : M = \mathcal{W}\mathbb{R}^2 \to \mathbb{R}^2$ are natural projections and

$$g_{[u]} = \epsilon\frac{1}{\sqrt{u}}\left(du^2 - \zeta dv^2\right)$$

with $\epsilon = \pm 1$. In the above construction one can substitute the quotient $\mathbb{R}^2/T$ for $\mathbb{R}^2$, where T denotes the discrete group acting on $\mathbb{R}^2$ generated by a transformation of the form $(\xi, \eta) \longrightarrow (\xi + a, \eta + a)$, $a \in \mathbb{R}$. Let now $(\mathcal{W}, u)$ be as before but w is a ζ-harmonic function on $\mathcal{W}$. Then $M = \mathcal{W}\mathbb{R}^2$ carries the Ricci-flat metric

$$g = \epsilon_1\pi_1^*\left(du^2 - \zeta^2 dv^2\right) + \epsilon_2\pi_2^*\left(d\xi^2 - d\eta^2\right) + \pi_1^*\left(w\right)\pi_2^*\left(\left(\frac{d\xi - d\eta}{\xi - \eta}\right)^2\right) \tag{10}$$

with $\epsilon_i = \pm 1$.

Theorem 5. *Any Ricci-flat 4-metric, admitting a non-extendible bidimensional non-commutative Killing algebra, is either of the form (9) or (10) with Killing leaves of one of two types* $\mathbb{R}^2$ *or* $\mathbb{R}^2/T$.

Proof. It follows directly from the local description of solutions given in proposition 10 and 13 of [4] and observations at the end of section *3* of [4]. □

Remark 4. Recall that geometric singularities of multivalued solutions are classified by types corresponding to finite dimensional commutative $\mathbb{R}$ algebras (see [5]). As it was noticed in the previous section, any global solution, described there in invariant terms, can be obtained from the model one (see next section) as its pullback via a ζ-holomorphic map of a ζ-complex $\mathcal{W}$ to $\mathbb{R}_\zeta$. Singularities of this map are interpreted naturally as geometric singularities of multivalued solutions of the Einstein equations over $\mathbb{R}_\zeta$. These singularities are either of type $\mathbb{C}$ or of type $\mathbb{R}^2 \oplus \mathbb{R}^2$. This shows that multivalued solutions of the Einstein equations admit such types of singularities and give further illustration of the theory (see [5]).

Remark 5. It is important to stress the existence of superposition laws for some classes of metrics. For instance, if in Case I one fixes an isomorphism class of *3*-dimensional Killing algebras, a ζ-complex curve $\mathcal{W}$, a fiber type and a constant A, then the corresponding solutions are parametrized by ζ-harmonic functions u on $\mathcal{W}$ and, therefore, constitute a linear space. If in Case II one fixes a ζ-harmonic functions u on $\mathcal{W}$, then the corresponding metrics are parametrized by solutions w of the linear equation $d\left(uJ^*\left(dw\right)\right) = 0$. Conversely, one can fix w and vary u.

Remark 6. Ricci flat metrics described in invariant terms in this section are *semi-global* in the sense that they are not extendible along the Killing leaves, but, generally, they are extendable along the orthogonal leaves. If the pair $(\mathcal{W}, u)$, over which a Ricci-flat metric of the considered type is constructed, is nonextendable, then the metric itself is also nonextendable. In this sense the results of this section give all global forms of the considered class of Ricci-flat metrics.

3 Model solutions, normal coordinates and the local geometry of leaves

In view of proposition *2* the basic geometrical properties of the metrics we are considering can be extracted from the model solutions. This is why they deserve special attention. Recall that by a *model solution* we understand one for which $(\mathcal{W}, u) = \left(\mathbb{R}^2_\zeta, x\right)$. Thus, coordinates (x, y), corresponding to the canonical ζ-complex variable $x + \zeta y$, are privileged in $\mathbb{R}^2_\zeta$ and also, *via* the pullback π_1^*, on orthogonal leaves. These coordinates will be used to introduce *normal coordinates* which are more convenient from a geometric view-point. It is worth to emphasize that ζ-complex coordinates (x, y), appearing automatically in our approach, generalize the well known Kruskal-Szekeres coordinates for the Schwarzschild metric. By their nature they do not depend on a choice of the branch of the function $(\beta - A)/\beta$. Metrics, we are analyzing, may be regarded as the *composition* of two families of bidimensional metrics, one along the Killing leaves and another along the orthogonal leaves. Then, the *composed* metrics can be characterized in terms of intrinsic and extrinsic peculiarities of these bidimensional metrics. **Killing Leaves**

They are homogeneous surfaces. Then, their intrinsic geometry is completely characterized by the value of the Gauss curvature. This value is a known function of u (of x for model solutions) and it remains to describe their extrinsic geometry, for instance their geodesic curvature. The geodesic flow, which corresponds to a non-commutatively integrable Hamiltonian vector field (proposition 2), projects on the geodesic flow of the metric restricted to a Killing leave. **Orthogonal leaves**

They have a trivial extrinsic geometry, namely, they are totally geodesic submanifolds (proposition 2). Thus, it remains to characterize their intrinsic geometry. To this purpose, notice that these leaves admit a Killing field, given in coordinates (u, v) ((x, y) for model solutions) by ∂_v. As it is well known, all scalar differential invariants of such bidimensional metrics are functions of their Gauss curvature K. Moreover, the metric itself is completely characterized by the function $I = I(K)$ for a single independent (for generic metrics) differential invariant I. For the model solution this function is given for $I = |\nabla K|^2$. The behavior of geodesics, which is particularly important from the physical view-point, is described below for metrics admitting a *3*-dimensional Killing algebra. Namely, we show that the corresponding geodesic flows are *non-commutatively integrable*, by explicitly exhibiting all necessary first integrals. The integrability of metrics with non-extendible Killing algebras is at the moment unclear. In what follows we report in a systematic way all the aforementioned data for the model solutions. Each case is labelled by a suitable subset of the following set of parameters $\left\{\mathcal{G}, \zeta^2, A, branch, \epsilon_1, \epsilon_2, \epsilon\right\}$, where ϵ_1, $\epsilon_2 = \pm 1$ are the constants in the equation (10) and $\epsilon = \pm 1$ is linked to the Gauss curvature K_0 of Killing leaves. Note that in the last case model solutions of the form (9), corresponding to a given label, are parametrized by solutions w of the equation

$$d\left(u\, J^*\left(dw\right)\right) = 0,$$

while those of the form (10) are parametrized by ζ -harmonic functions on $\mathbb{R}^2_\zeta$.

It is easy to see that metrics corresponding to different labels, or different parametrizing functions are not *isometric*.

3.1 $\mathcal{G} = so\left(2,1\right),\ \mathcal{G} = so(3)$

In this case the model solutions are either flat metrics ($A = 0$) or have the following local expression in terms of the canonical coordinates (x, y) on $\mathcal{W}$ (see (5))

$$g = \epsilon_1 \frac{\beta - A}{\beta}\left(dx^2 - \zeta^2 dy^2\right) + \epsilon_2 \beta^2\left(d\vartheta^2 + F\left(\vartheta\right) d\varphi^2\right) \tag{11}$$

where $F\left(\vartheta\right)$ is equal, in the case of $so\left(2,1\right)$, either to $\sinh^2\vartheta$ or to $-\cosh^2\vartheta$, depending on the signature of the metric, and to $\sin^2\vartheta$ in the case of $so\left(3\right)$. The function $\beta\left(x\right)$ is defined by one of the positive branches of the tortoise equation

$$\beta + A\log|\beta - A| = x$$

with $A \neq 0$. By introducing the *normal coordinates* $(r = \beta, \tau = y, \vartheta, \varphi)$, the local expression of g becomes

$$g = \epsilon_1 \left(\frac{r}{r-A} dr^2 - \zeta^2 \frac{r-A}{r} d\tau^2 \right) + \epsilon_2 r^2 \left(d\vartheta^2 + F(\vartheta)\, d\varphi^2 \right) \tag{12}$$

3.1.1 Killing leaves

The Gaussian curvature K. In the cases $so(2,1)$ or $so(3)$ it is given by

$$K_{so(2,1)} = -\frac{\epsilon_2}{r^2}$$
$$K_{so(3)} = \frac{\epsilon_2}{r^2},$$

respectively. Thus, as already mentioned in section 3 of [4], we see that the Killing leaves are bidimensional Riemannian manifolds, with constant Gauss curvature whose value, depending on r, changes with the leaf. In the case of $so(2,1)$ the Killing leaves are *non-Euclidean* or *"anti" non-Euclidean* planes, depending on whether ϵ_2 is positive or negative, assuming that $F(\vartheta) = \sinh^2 \vartheta$, while they have an indefinite metric with nonvanishing constant Gauss curvature if $F(\vartheta) = -\cosh^2 \vartheta$.

In the case of $so(3)$, the Killing leaves are standard (metric) spheres or "anti"-spheres, depending on whether ϵ_2 is positive or negative.

The second fundamental form II We evaluate its components with respect to normal unit vector fields parallel to the coordinate fields, namely

$$II(X,Y) = (\nabla_X Y, n_r)\, n_r + (\nabla_X Y, n_y)\, n_y,$$

where X and Y are tangent to the Killing leaves, and $n_i = \sqrt{|g^{ii}|}\partial_i$; here the index i is either r or y. In the normal coordinate the associated matrices $II^y$and II^r are

$$(II^r_{ab}) = \sqrt{|g_{rr}|}\, (\Gamma^r_{ab}) = -\frac{\epsilon_1}{r} \left(\frac{r-A}{r} \right) \sqrt{\left| \left(\frac{r}{r-A} \right) \right|} \mathbf{H}$$
$$(II^y_{ab}) = \sqrt{|g_{yy}|}\, (\Gamma^y_{ab}) = 0,$$

where $\mathbf{H}$ is the lower block of the matrix associated to g in the normal coordinates, and the Γ's are the Christoffel symbols of g.

The Christoffel symbols of the normal connection $\widetilde{\Gamma}^j_{ai}$ Since the normal coordinates are adapted coordinates, the Christoffel symbols are defined by

$$\widetilde{\Gamma}^j_{ai}\partial_j = \nabla_a \left(\partial_i\right)^{\perp},$$

so that

$$\widetilde{\Gamma}^j_{ai} = \Gamma^j_{ai} = 0.$$

The geodesic curvature of Killing leaves Recall that the *geodesic curvature* C_g of a curve on M whose parametric equations are given by $x_\mu(s)$, where s is a *natural parameter*, is given by

$$C_g^2 = g_{\mu\nu}\Phi^\mu\Phi^\nu$$

with $\Phi^\lambda \equiv \dot{V}^\lambda + \Gamma^\lambda_{\mu\nu}V^\mu V^\nu$, where $V^\mu = \dot{x}_\mu$ are the components of the tangent vector to the curve. Observe that the curve, defined by $\varphi = 0$ on a Killing leaf, is a geodesic for the metric restricted to a Killing leaf. In normal coordinates this geodesic γ, considered as a curve on M, is given by

$$r = const., \quad y = 0, \quad \vartheta = \frac{s}{r}, \quad \varphi = 0$$

so that

$$V^r = 0, \quad V^y = 0, \quad V^\vartheta = \frac{1}{r}, \quad V^\varphi = 0,$$

and

$$\Phi^\lambda = \Gamma^\lambda_{\vartheta\vartheta} = \delta_{\lambda r}\Gamma^r_{\vartheta\vartheta} = -\epsilon_1\delta_{\lambda r}\left(r - A\right).$$

Therefore, we obtain

$$C_g^2 = \epsilon_1 r\left(r - A\right). \tag{13}$$

By obvious symmetry arguments the geodesic curvature of any geodesic on a Killing leaf with respect to the induced metric is given by (13). So, this quantity characterizes completely the geodesic curvature of the Killing leaves.

3.1.2 Orthogonal leaves

The scalar differential invariants The Gaussian curvature K and $|\nabla K|^2$are

$$K = \frac{\epsilon_1 A}{r^3}$$
$$|\nabla K|^2 = 9K^3\left(1 + \left(\frac{1}{A^2K}\right)^{\frac{1}{3}}\right).$$

The Christoffel symbols of the normal connection $\widetilde{\Gamma}^b_{ia}$ They are given by

$$\widetilde{\Gamma}^b_{ra} = \Gamma^b_{ra} = \delta_{ab}\frac{1}{r}, \quad \widetilde{\Gamma}^b_{ya} = \Gamma^b_{ya} = 0.$$

3.2 $\mathcal{G}$=$\mathcal{K}$*il*$(d\xi^2 \pm d\eta^2)$

In this case the model solutions are either flat metrics or have the following local expression in terms of the coordinates (x, y) which in this case are also normal coordinates (see (8)):

$$g = \epsilon_1 \frac{1}{\sqrt{x}}\left(dx^2 - \zeta^2 dy^2\right) + \epsilon_2 x\left(d\xi^2 + \epsilon d\eta^2\right)$$

where $\epsilon_i = \pm 1$.

3.2.1 Killing leaves

They are flat 2-manifolds.

The second fundamental form II As before, we evaluate its components with respect to normal unit vector fields parallel to the coordinate fields. The matrices $II^x$and II^y are given by

$$(II^x_{ab}) = \sqrt{|g_{xx}|}\,(\Gamma^x_{ab}) = -\frac{\epsilon_1}{2}\,|x|^{-\frac{3}{4}}\,\mathbf{H}$$
$$II^y_{ab} = \sqrt{|g_{yy}|}\Gamma^y_{ab} = 0,$$

where $\mathbf{H}$ is the lower block of the matrix associated to g in the normal coordinates, and Γ's are the Christoffel symbols of g.

The Christoffel symbols of the normal connection They are given by

$$\widetilde{\Gamma}^j_{ai} = \Gamma^j_{ai} = 0$$

The geodesic curvature of the Killing leaves As before, in the case of non null geodesics, it is characterized by

$$C^2_g = \epsilon_1 \frac{x^{-\frac{3}{4}}}{4}.$$

3.2.2 Orthogonal leaves

The scalar differential invariants The Gaussian curvature K and $|\nabla K|^2$ are given by

$$\begin{aligned} K &= -\frac{\epsilon_1}{4x^{\frac{3}{2}}} \\ |\nabla K|^2 &= -9K^3 \end{aligned}$$

The Christoffel symbols of the normal connection They are given by $\widetilde{\Gamma}^b_{ia}$. Thus, we have

$$\begin{aligned} \widetilde{\Gamma}^b_{\rho a} &= \Gamma^b_{\rho a} = \frac{1}{2}\delta_{ab} \\ \widetilde{\Gamma}^b_{\sigma a} &= \Gamma^b_{\sigma a} = 0. \end{aligned}$$

3.3 Non-extendible bidimensional non-commutative Killing algebra

In this case, the coordinates (x, y) are also normal coordinates, and the model solutions have one of the following local expressions (see (9), (10)):

$$g = \epsilon_1 \frac{1}{\sqrt{x}} \left(dx^2 - \zeta^2 dy^2\right) + \epsilon_2 x \left[\left(d\xi^2 - d\eta^2\right) + w\left(\frac{d\xi - d\eta}{\xi - \eta}\right)^2\right], \tag{14}$$

where w is a solution of the equation

$$x\left(\frac{\partial^2}{\partial x^2} - \zeta^2 \frac{\partial^2}{\partial y^2}\right) w + \frac{\partial w}{\partial x} = 0;$$

or

$$g = \epsilon_1 \left(dx^2 - \zeta^2 dy^2\right) + \epsilon_2 \left[\left(d\xi^2 - d\eta^2\right) + w\left(\frac{d\xi - d\eta}{\xi - \eta}\right)^2\right] \tag{15}$$

where w is a ζ-harmonic function.

3.3.1 Killing leaves

They are flat manifolds.

The second fundamental form II As before, we evaluate its components with respect to normal unit vector fields parallel to the coordinate fields. The matrices II^x and II^y for the metric

(14), are given by

$$(II^x_{ab}) = \sqrt{|g_{xx}|}\,(\Gamma^x_{ab}) = -\frac{\epsilon_1\epsilon_2}{2}\,|x|^{-\frac{3}{4}}\left[(g_{ab}) + x^2\frac{\partial_x w}{(\xi-\eta)^2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\right]$$
$$(II^y_{ab}) = \sqrt{|g_{yy}|}\,(\Gamma^y_{ab}) = \frac{\epsilon_1\epsilon_2\zeta^2}{2}\,|x|^{\frac{5}{4}}\,\frac{\partial_y w}{(\xi-\eta)^2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

while for the metric (15) one has

$$(II^x_{ab}) = \sqrt{|g_{xx}|}\,(\Gamma^x_{ab}) = -\frac{\epsilon_1\epsilon_2}{2}\frac{\partial_x w}{(\xi-\eta)^2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$
$$(II^y_{ab}) = \sqrt{|g_{yy}|}\,(\Gamma^y_{ab}) = \frac{\epsilon_1\epsilon_2\zeta^2}{2}\frac{\partial_y w}{(\xi-\eta)^2}\,(\xi-\eta)^2\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

The Christoffel symbols of the normal connection:

$$\widetilde{\Gamma}^j_{ai} = \Gamma^j_{ai} = 0$$

The geodesic curvature of Killing leaves. The geodesic curvature for the metric (14) is given by

$$C_g^2 = \frac{\epsilon_1\,|x|^{-\frac{3}{2}}}{4}\left[(1 + xV\partial_x w)^2 - (xV\partial_y w)^2\right],$$

where $V = \epsilon_2\left(\frac{x}{(\xi-\eta)^2}\right)\left(V^\xi - V^\eta\right)^2$ with V^ξ and V^η components of a vector tangent to a geodesic of the restricted metric. Their expression is

$$V^\xi = \frac{1}{2}\frac{(a+k)(\xi-\eta) - 3kw}{\sqrt{\left|kx\left[a(\xi-\eta)^2 - 2wk\right]\right|}}$$
$$V^\eta = \frac{1}{2}\frac{(a-k)(\xi-\eta) - 3kw}{\sqrt{\left|kx\left[a(\xi-\eta)^2 - 2wk\right]\right|}}$$

with k and a arbitrary constants. For the metric (12) the geodesic curvature is

$$C_g^2 = \frac{\epsilon_1}{4}\left[1 + \left(\frac{V^\xi - V^\eta}{\xi-\eta}\right)^4\left[(\partial_x w)^2 - (\partial_y w)^2\right] + 2\epsilon_2\left(\frac{V^\xi - V^\eta}{\xi-\eta}\right)^2\partial_x w\right]$$

3.3.2 Orthogonal leaves

The scalar differential invariants. The Gaussian curvature K and $|\nabla K|^2$ are given by

$$K = -\frac{\epsilon_1}{4x^{\frac{3}{2}}}$$
$$|\nabla K|^2 = -9K^3$$

The second fundamental form II As before, we evaluate its components with respect to normal unit vector fields parallel to the coordinate fields, so that

$$II^a_{ij} = \Gamma^a_{ij}\sqrt{|g_{aa}|} = 0.$$

The Christoffel symbols of the normal connection $\widetilde{\Gamma}^b_{ia}$ For the metric (14) they are

$$\begin{aligned}\left(\widetilde{\Gamma}^b_{xa}\right) &= \left(\Gamma^b_{xa}\right) = \frac{1}{2x}\left(\delta_{ba}\right) + \frac{\epsilon_2}{2}\frac{\partial_x \widetilde{w}}{(\xi-\eta)^2}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\\ \left(\widetilde{\Gamma}^b_{ya}\right) &= \left(\Gamma^b_{ya}\right) = \frac{\epsilon_2}{2}\frac{\partial_y \widetilde{w}}{(\xi-\eta)^2}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix},\end{aligned}$$

while for the metric (15)

$$\left(\widetilde{\Gamma}^b_{ia}\right) = \left(\Gamma^b_{ia}\right) = \frac{\epsilon_2}{2}\frac{\partial_i \widetilde{w}}{(\xi-\eta)^2}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \qquad i = x, y.$$

In the above matrices b is the row index and a the column index:

3.4 Geodesic flows

Geodesic flows corresponding to the metrics we are dealing with show some interesting properties, one of which will be used in next section when describing the info-hole phenomenon. Below, having this application in mind, we shall discuss briefly only the flows, corresponding to the metrics, admitting a 3-dimensional Killing algebra. Recall that the geodesic flow, corresponding to a metric $g = g_{ij}dx_i dx_j$ on $\mathcal{M}$, can be viewed as the flow, generated by the Hamiltonian field X_H, $H = \frac{1}{4}g^{ij}p_i p_j$ (p_i's are conjugated to x_i's coordinates on $T^*\mathcal{M}$). For example, up to the factor $\frac{1}{4}$, the Hamiltonian

$$\mathcal{H} = \epsilon_1\left[\left(1-\frac{A}{r}\right)p_r^2 - \zeta^2\frac{p_\tau^2}{1-\frac{A}{r}}\right] + \frac{\epsilon_2}{r^2}\left(p_\vartheta^2 + G\left(\vartheta\right)p_\varphi^2\right), \tag{16}$$

corresponds to the metric (13) expressed in the normal coordinates (see subsection 3.1). Notice also that the projection $\pi_2 : M = \mathcal{W}\Sigma \to \Sigma$ generates canonically a projection

$$\bar{\pi}_2 : T^*M \to T^*\Sigma.$$

Proposition 6. *Projection π_2 sends geodesic (nonparametrized) curves of a model Ricci-flat metrics associated with a 3-dimensional Killing algebra to geodesic (nonparametrized) curves of the metric g_Σ.*

Proof. It follows from the relation

$$d_a \bar{\pi}_2 \left(X_{H,a}\right) = \lambda X_{H_{\Sigma},\bar{\pi}_2(a)} \qquad \lambda \in \mathbb{R},\, a \in T^*M,$$

which, in its turn, is a direct consequence of the particular form of the Hamiltonian (16) (and similarly for algebras $Kil\left(dx^2 + dy^2\right)$. ◻

It is worth to note that geodesic flows, corresponding to the model Ricci-flat metrics, associated with 3-dimensional Killing algebras, are integrable. To see that one may observe that, for instance, Hamiltonian (16) with $G(\vartheta) = \frac{1}{\sinh^2 \vartheta}$ possesses five independent first integrals

$$\mathcal{H}, \quad p_\tau, \quad p_\vartheta^2 + \frac{1}{\sinh^2 \vartheta} p_\varphi^2, \quad \mathcal{I}_1, \quad \mathcal{I}_2$$

where $\mathcal{I}_1$ and $\mathcal{I}_2$ are generators of a noncommutative bidimensional subalgebra of $so(2,1)$, for example

$$\begin{aligned}
\mathcal{I}_1 &= \left[\left(1+\sqrt{2}\right)\cos\varphi + \sin\varphi\right] p_\vartheta + \left[1 + \sqrt{2} + \coth\vartheta\left(\cos\varphi - \left(1+\sqrt{2}\right)\sin\varphi\right)\right] p_\varphi \\
\mathcal{I}_2 &= \sqrt{2}\left[\cos\varphi + \sin\varphi\right] p_\vartheta + \left[2 + \sqrt{2}\coth\vartheta\left(\cos\varphi - \sin\varphi\right)\right] p_\varphi.
\end{aligned}$$

The 5 first integrals span a rank 3 Lie algebra $\mathcal{A}$, and since

$$\operatorname{rank}\mathcal{A} + \dim\mathcal{A} = \dim T^*M \quad \text{and} \qquad \operatorname{rank}\mathcal{A} \; < \dim\mathcal{A}$$

the system is noncommutatively integrable in the sense [1, 3]. Note also that the above proposition is no more valid for the geodesic flow of Ricci-flat metrics with nonextendable bidimensional Killing algebras.

3.5 The geodesic flow (non-extendible case)

Here, apart from the solution $w = const$, the geodesic curves do not project on the Killing fields and $h^{ab}p_a p_b$ is no more a first integral for the geodesic equations.

4 Examples

In this concluding section we illustrate the previous general results with a few examples. According to proposition *2*, we can construct any solution as the pullback of a model solution *via* a ζ-holomorphic map Φ of a ζ-complex curve $\mathcal{W}$ to $\mathbb{R}^2_\zeta$. Recall that in the pair $(\mathcal{W}, u)$, describing the

so obtained solution, $u = \mathrm{Re}\ \Phi$. A detailed analysis of geometrical properties of the obtained exact solutions, as well as their possible physical interpretation, is postponed up in a forthcoming paper.

4.1 *Algebraic solutions*

Let $\mathcal{W}$ be an algebraic curve over $\mathbb{C}$, understood as a ζ-complex curve with $\zeta^2 = -1$. With a given meromorphic function Φ on $\mathcal{W}$ a pair $(\mathcal{W}_\Phi, u)$ is associated, where $\mathcal{W}_\Phi$ is $\mathcal{W}$ deprived of the poles of Φ and u the real part of Φ. A solution (metric) constructed over such a pair will be called *algebraic*. Algebraic metrics are generally singular. For instance, such a metric is degenerate along the fiber $\pi_1^{-1}(a)$ (see section *2*) if $a \in \mathcal{W}$ is such that $d_a u = 0$.

4.2 *Info-holes*

Space-times corresponding to algebraic metrics and, generally, to metrics with signature equal to 2 constructed over complex curves ($\zeta^2 = -1$), exhibit the following interesting property: *for a given observer there exists another observer which can be never contacted*. By defining an *info-hole* (information-hole) of a given point a to be the set of points of the space-time whose future does not intersect the future of a, the above property can be paraphrased by saying that *the info-hole of a given point of such a space-time is not empty*. In fact, consider a metric of the form (5) constructed over a complex curve whose standard fiber Σ is a bidimensional manifold supplied with an indefinite metric of constant Gauss curvature equal to 1. For our purpose, it is convenient to take for Σ the hyperboloid $x_1^2 + x_2^2 - x_3^2 = 1$, supplied with the induced metric $g|_\Sigma = dx_1^2 + dx_2^2 - dx_3^2|_\Sigma$. The *light-cone* of $g|_\Sigma$ at a given point b is formed by the pair of rectilinear generators of Σ passing through b. Since the geodesics of g project *via* π_2 into geodesics of $g|_\Sigma$ (see section *3*), it is sufficient to prove the existence of info-holes for the bidimensional space-time $(\Sigma, g|_\Sigma)$. To this purpose, consider the standard projection π of $\mathbb{R}^3 = \{(x_1, x_2, x_3)\}$ onto $\mathbb{R}^2 = \{(x_1, x_2)\}$. Then, $\pi|_\Sigma$ projects Σ onto the region $x_1^2 + x_2^2 \geq 1$ in $\mathbb{R}^2$ and the rectilinear generators of Σ are projected onto tangents to the circle $x_1^2 + x_2^2 = 1$. Suppose that the time arrow on Σ is oriented according to increasing value of x_3. Then the future region $F(b) \subset \Sigma$ of the point $b \in \Sigma$, $b(1, 1, \beta)$ $\beta > 0$, projects onto the domain defined by $D = \{x \in \mathbb{R}^2 : x_1 > 1, x_2 > 1\}$, and the future region of any point $b' \in \Sigma$, such that $\pi(b') \in D'$ and $x_3(b') > 0$, does not intersect $F(b)$. By obvious symmetry

arguments the result is valid for any point $b \in \Sigma$

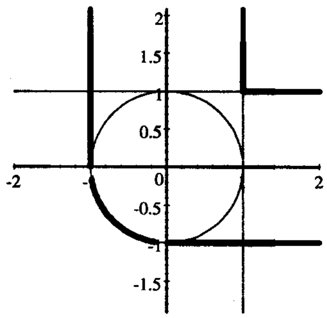


*Fig.*2

4.3 *A star "outside" the universe*

The Schwarzschild solution shows a "star" generating a space "around" itself. It is an $so(3)$-invariant solution of the vacuum Einstein equations. On the contrary, its $so(2,1)$-analogue (see section *3) shows a "star" generating the space only on "one side of itself". More exactly, the fact that the space in the Schwarzschild universe is formed by a 1-parametric family of "concentric" spheres allows one to give a sense to the adverb "around". In the* $so(2,1)$-case the space is formed by a 1-parameter family of "concentric" hyperboloids. The adjective "concentric" means that the curves orthogonal to hyperboloids are geodesics and metrically converge to a singular point. This explains in what sense this singular point generates the space only on "one side of itself".

4.4 *Kruskal-Szekeres type solutions*

We describe now a family of solutions which are of the *Kruskal-Szekeres type* [6], namely, that are characterized as being maximal extensions of the local solutions determined by an affine parametrization of null geodesics, and also by the use of more than one interval of monotonicity of $u(\beta)$. Consider the ζ-complex curve

$$\mathcal{W} = \left\{ (z = x + \zeta y) \in \mathbb{R}^2_\zeta : y^2 - x^2 < 1 \right\}, \ \ \zeta^2 = 1$$

and the ζ-holomorphic function $\Phi : \mathcal{W} \to \mathbb{R}^2_\zeta$

$$\Phi(z) = A \ln\left(|A|\, z^2\right) = A\left(\ln\left|A\left(x^2 - y^2\right)\right| + \zeta \ln\left|\frac{x+y}{x-y}\right|\right).$$

Thus, in the pair $(\mathcal{W}, u)$ the ζ-harmonic function u is given by

$$u = A \ln \left| A \left(x^2 - y^2\right) \right| .$$

Let us decompose $\mathcal{W}$ in the following way:

$$\mathcal{W} = \mathcal{U}_1 \cup \mathcal{U}_2$$

where

$$\begin{aligned} \mathcal{U}_1 &= \left\{ (z = x + \zeta y) \in \mathbb{R}^2_\zeta : 0 \leq y^2 - x^2 < 1 \right\} \\ \mathcal{U}_2 &= \left\{ (z = x + \zeta y) \in \mathbb{R}^2_\zeta : y^2 - x^2 \leq 0 \right\} . \end{aligned}$$

Consider now the solution defined as the pull back with respect to $\Phi|_{\mathcal{U}_1}$ and $\Phi|_{\mathcal{U}_2}$ of the model solutions determined by the following data: in the case of $\Phi|_{\mathcal{U}_1}$, $\mathcal{G} = so\,(3)$ or $\mathcal{G} = so\,(2,1)$, (see equation (11)), characterized by $F\,(\vartheta) = sin^2\vartheta$ or $F\,(\vartheta) = sinh^2\vartheta$ respectively, $\epsilon_1 = \epsilon_2 = 1$, $A > 0$, and for $\beta\,(u)$ the interval $]0, A]$; in the case of $\Phi|_{\mathcal{U}_2}$ the same data except for $\beta\,(u)$ which belongs to the interval $[A, \infty[$. The case $F\,(\vartheta) = sin^2\vartheta$, corresponding to $so\,(3)$, will give the Kruskal-Szekeres solution (see [6]). The case $F\,(\vartheta) = sinh^2\vartheta$, corresponding to $so(2,1)$, will differ from the previous one in the geometry of the Killing leaves, which will now have a negative constant Gaussian curvature. The metric g has the following local form

$$g = 4A^3 \frac{\exp \frac{\beta}{A}}{\beta} \left(dy^2 - dx^2\right) + \beta^2 \left[d\vartheta^2 + F\,(\vartheta)\, d\varphi^2\right] ,$$

the singularity $\beta = 0$ occurring at $y^2 - x^2 = 1$.

4.5 *The "square root" of the Schwarzschild universe*

Now we will discuss Einstein metrics of signature 2, induced by a ζ-*quadratic* map, $\zeta^2 = 1$, from a model of the Schwarzschild type. The Einstein manifolds obtained in this way are interpreted naturally as *parallel universes* generated by *parallel "stars".*

1. Further on it is assumed that $\zeta^2 = 1$. Recall that a model metric of Schwarzschild type is characterized by the following data:

$$\mathcal{G} = so\,(3) \;\text{ or } \mathcal{G} = so\,(2,1)\,,\; \epsilon_1 = \epsilon_2 = 1,\; A > 0,\; \beta\,(u) \in [A, \infty[$$

Its *square root* is the metric induced from it by the ζ-*quadratic* map $z \longmapsto \frac{\zeta}{2} z^2$ of $\mathbb{R}^2_\zeta$ into itself. Obviously, the basic pair $(\mathcal{W}, u)$ determining the square root is $\left(\mathbb{R}^2_\zeta, xy\right)$, $\quad z = x + \zeta y$,

The local expression of the metric is

$$g=-\frac{1}{2}\frac{\exp\frac{\beta+xy}{A}}{\beta}\left(y^2-x^2\right)\left(dy^2-dx^2\right)+\beta^2\left[d\vartheta^2+F\left(\vartheta\right)d\varphi^2\right] \tag{17}$$

where F, depending on the Gauss curvature K, is one of the functions $\sin^2\vartheta$, $\sinh^2\vartheta$, $-\cosh^2\vartheta$ according to equation (11). The metric is degenerate along the lines $y=\pm x$. For our purposes it is convenient to refer the *square root* metric to the Kruskal-Szekeres type coordinates considered in the previous example. To this end consider the holomorphic map

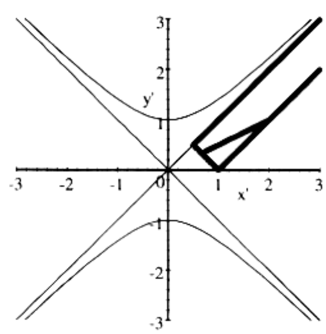
$$\mathcal{F}:\mathbb{R}^2_\zeta\longrightarrow\mathbb{R}^2_\zeta$$

$$\mathcal{F}\left(z\right)=\frac{1}{\sqrt{|A|}}\exp\frac{\zeta z^2}{4A}.$$

Denote the quadrants of the ζ-complex line $z=x+\zeta y$ bounded by the lines $x=\pm y$ as

$$I=\left\{x\geq|y|\right\},\quad II=\left\{y\geq|x|\right\},\quad III=\left\{-x\geq|y|\right\},\quad IV=\left\{-y\geq|x|\right\}.$$

$\mathcal{F}$ maps each of them onto the strip in the ζ-complex line $z'=x'+\zeta y'$ enclosed by the thick lines as it is shown in the figure for $A=1$.



x',y'
$Fig.3$

The image of the lines $x=\pm y$ are respectively the half line a and the segment b. The third side of the strip corresponds to $x-y=\pm\infty$. In *normal coordinates* $r=\beta,\quad\tau=x^2+y^2$, a fixed value of τ determines a segment in each region (the thick segment c in the figure). Its pull back with respect to π_1 (see section 2.2) is a spherical or hyperbolic shell for $\mathcal{G}=so\left(3\right)$ or $\mathcal{G}=so\left(2,1\right)$, respectively, (see example 3). Since the Einstein metrics we are considering is a pull back of a Schwarzschild type one it comes natural to interpret the maximal radius of each of these shells as

the dimension of the universe at the corresponding instant of time, while the minimal radius gives the instant value of the Schwarzschild horizon. The shells corresponding to quadrants I and II are soldered along their maximal radius spheres (respectively, hyperboloids) forming one, say $\mathcal{U}_1$, of two *parallel universes*. Another one, $\mathcal{U}_2$, is related similarly with quadrants III and IV. These two universes have two common Schwarzschild horizons along which the shells corresponding to quadrants II, III and I, IV , respectively, are soldered. Associating a "star" with each of these two horizons one discovers a system of two parallel stars in a perfect equilibrium, which generate two *parallel universes,* $\mathcal{U}_1$ and $\mathcal{U}_2$ whose dimensions grow infinitely with time. **2**. As in example 4 the metric we are considering can be extended beyond the Schwarzschild horizon $x = y$. To this end consider

$$\begin{aligned} \mathcal{W}_1 &:= \left\{ z = x + \zeta y \in \mathbb{R}^2_\zeta : |x| < |y| \Longrightarrow \; yx < A \ln A \right\}, \;\; \zeta^2 = 1 \\ \Phi\left(z\right) &= \frac{\zeta}{2} z^2 = xy + \frac{1}{2}\zeta\left(x^2 + y^2\right). \end{aligned}$$

Thus, in the basic pair $(\mathcal{W}_1, u)$ the ζ-harmonic function u is given by

$$u = xy.$$

Let us decompose $\mathcal{W}_1$ in the following way:

$$\mathcal{W}_1 = \mathcal{V}_1 \cup \mathcal{V}_2$$

where

$$\begin{aligned} \mathcal{V}_1 &= \left\{ (z = x + \zeta y) \in \mathbb{R}^2_\zeta : |x| \le |y|, \;\; yx < A \ln A \right\} \\ \mathcal{V}_2 &= \left\{ (z = x + \zeta y) \in \mathbb{R}^2_\zeta : |x| \ge |y| \right\}. \end{aligned}$$

Consider now the solution defined as the pull back with respect to $\Phi|_{\mathcal{U}_1}$ and $\Phi|_{\mathcal{U}_2}$ of the model solutions determined by the following data: in the case of $\Phi|_{\mathcal{V}_1}$

$$\mathcal{G} = so\left(3\right) \;\text{ or } \mathcal{G} = so\left(2,1\right), \; \epsilon_1 = \epsilon_2 = 1, \; A > 1, \; \beta\left(u\right) \in \left]0, A\right],$$

in the case of $\Phi|_{\mathcal{V}_2}$

$$\mathcal{G} = so\left(3\right) \;\text{ or } \mathcal{G} = so\left(2,1\right), \; \epsilon_1 = \epsilon_2 = 1, \; A > 1, \; \beta\left(u\right) \in \left[A, \infty\right[.$$

In this case a branch change occurs along the singular lines $y = \pm x$ which become also *first-type discontinuity* lines for β. In the above local expression there is a singularity at $\beta = 0$, which is also

a divergence of the scalar curvature. Even in this case it is convenient to refer the metric to the Kruskal-Szekeres type coordinates. In the case $|x| > |y|$ consider the holomorphic map

$$z' = \frac{1}{\sqrt{|A|}} \exp \frac{\zeta z^2}{4A} \qquad z' = x' + \zeta y'$$

or,

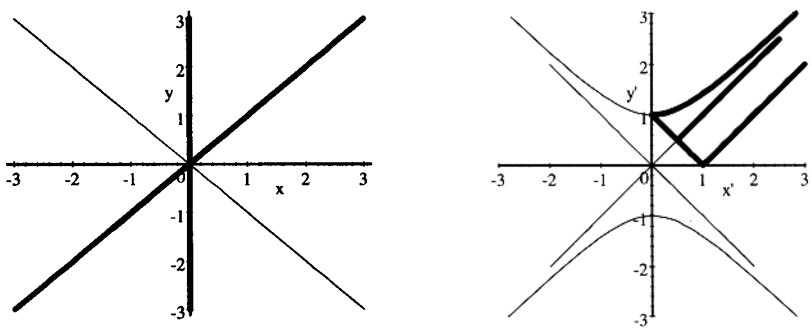
$$\begin{cases} U = y' - x' = -\frac{1}{\sqrt{|A|}} \exp -\frac{1}{4A} (y-x)^2 \\ V = y' + x' = \frac{1}{\sqrt{|A|}} \exp \frac{1}{4A} (y+x)^2 , \end{cases}$$

while, for $|x| < |y|$ and $yx < A \ln A$ consider the map

$$z' = \frac{\zeta}{\sqrt{|A|}} \exp \frac{\zeta z^2}{4A}$$

or,

$$\begin{cases} U = y' - x' = \frac{1}{\sqrt{|A|}} \exp -\frac{1}{4A} (y-x)^2 \\ V = y' + x' = \frac{1}{\sqrt{|A|}} \exp \frac{1}{4A} (y+x)^2 , \end{cases}$$



x, y $\qquad$ x', y'

*Fig.*4

From the above figures, which correspond to the map with $A = 1$, one can see that the region defined by $|x| > |y|$ is mapped onto the lower strip. The region defined by $|x| < |y|$ and $yx < A \ln A$ is mapped onto the upper strip and thus has the singularity at $\beta = 0$ as part of its boundary. For this region the line $x = y$, on which in the previous example occurred the soldering of the shells ($\tau =$ const.) along the maximal radius, now lies beyond the singularity. The lines $x - y = \pm\infty$ are mapped into the line $x' = y'$.

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